Chapter 2 Categories

§2.3 Products, Coproducts, and Universal Constructions

- Product
- Coproduct
- Free objects, initial and terminal objects, pull-back, push-out.

(I) Product

<u>Def</u> 3.1 For $\{A_i\}_{i\in I}$ in a given category $\mathcal{C}$, the products is defined as $(P, \{\pi_i\})$ with $P \in Ob\,\mathcal{C}$ and $\pi_i : P \longrightarrow A_i$ a family of maps such that for any $(B, \{\varphi_i\})$ with $\varphi_i : B \longrightarrow A_i$, there exists a unique $\varphi : B \longrightarrow P$ s.t. $\pi_i \varphi = \varphi_i$, $\forall\, i \in I$

$$\begin{tikzcd} B \arrow[d, dashed, "\exists!\varphi"'] \arrow[dr, "\varphi_i"] & \\ P \arrow[r, "\pi_i"'] & A_i \end{tikzcd}$$

We denote the product as $\prod_{i\in I} A_i$.

<u>Prop</u> 3.1 If $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both product of $\{A_i\}_{i\in I}$, then they are isomorphic.

Proof. For each product, we have

$$\begin{tikzcd} P \arrow[r, "\pi_i"] & A_i \\ Q \arrow[u, dashed, "\exists!\psi"] \arrow[ur, "\psi_i"'] & \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\psi_i"] & A_i \\ P \arrow[u, dashed, "\exists!\pi"] \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

Then

$$\begin{tikzcd} P \arrow[r, "\pi_i"] & A_i \\ P \arrow[u, "\psi\circ\pi"] \arrow[ur, "\pi_i"'] & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\pi_i"] & A_i \\ P \arrow[u, "id_P"] \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

Uniqueness implies $\psi\circ\pi = id$. Similarly $\pi\circ\psi = id$.

Remark. Generally, for $\{A_i\}_{i\in I} \subseteq \mathrm{Ob}\,\mathcal{C}$, there may not exist a product. But for Abelian category, the product and coproduct always exist.

(II) Coproduct

Coproduct is dual concept to product.

Def 3.2 For $\{A_i\}_{i\in I} \subseteq \mathrm{Ob}\,\mathcal{C}$, its coproduct is defined as $(S, \{\iota_i : A_i \to S\})$ which satisfies: for any $(B, \{\psi_i : A_i \longrightarrow B\})$, there exists a unique ψ s.t. $\psi \iota_i = \psi_i$:

$$\begin{tikzcd} A_i \arrow[r, "\iota_i"] \arrow[dr, "\psi_i"'] & S \arrow[d, dashed, "\exists!\psi"] \\ & B \end{tikzcd}$$

We denote the coproduct as $\coprod_{i\in I} A_i$.

Prop 3.2. If $(S, \{\iota_i\})$, $(S', \{\iota_i'\})$ are both coproducts of $\{A_i\}_{i\in I}$, then they are isomorphic.

Proof. Similar to Prop 3.1

Example 3.1. In module category Mod_R, product and coproduct are direct product and direct sum.

Example 3.2. In Set: (1) product = Cartesian product

(2) coproduct = disjoint union

(III) Some special objects.

1. Free object.

Def 3.3 A concrete category $\mathcal{C}$ is a category that is equipped with a (faithful) functor to the Set category. More precisely:

(1) Every object A is assigned with a set $\sigma(A)$.

(2) Every map $A \xrightarrow{f} B$ is assigned with a set map $\sigma(A) \longrightarrow \sigma(A)$.

(3) $id_A = id_{\sigma(A)}$

(4) $\sigma(f \circ g) = \sigma(f) \circ \sigma(g)$

$$A \xrightarrow{g} B \xrightarrow{f} \quad \rightsquigarrow \quad \sigma(A) \xrightarrow{\sigma(f)} \sigma(B) \xrightarrow{\sigma(g)} \sigma(C)$$

Remark. In concrete category, we can regard object as a set equipped with some additional structure.

Example: Grp, Ring, Vect, Mod_R are all concrete categories.

Recall that for free module M and its basis X, consider the inclusion $\iota : X \hookrightarrow M$, then for any module N and set map $f : X \longrightarrow N$, there exists unique $\tilde{f}$ s.t. $f = \tilde{f} \circ \iota$.

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & M \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & N \end{tikzcd}$$

This inspire the following definition of free objects

Def 3.4 Let V be an object of some concrete category $\mathcal{C}$, $i : X \longrightarrow V$ is a set map. If for any object $A \in Ob\,\mathcal{C}$ and set map $f : X \longrightarrow A$, there is a unique $\tilde{f} : V \to A$ s.t. $f = \tilde{f} \circ i$, then V is called a free object over X.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & V \arrow[d, dashed, "\exists \tilde{f}"] \\ & A \end{tikzcd}$$

Example. free module is free object in Mod_R.

Prop 3.3 Let $\mathcal{C}$ be a concrete category and V be a free object over X, V' be a free object over X', if $|X| = |X'|$, then $V \cong V'$.

Proof. Exercise.

2. Initial and terminal object.

means $A \xrightarrow{f} B$ exists & unique

Def 3.5 For a category $\mathcal{C}$

(1) A is called an *initial object* if for any B, $\# \mathrm{Hom}(A, B) = 1$.

(2) A is called a terminal object if for any B, $\# \mathrm{Hom}(B, A) = 1$.

(3) A is called a zero or null object if it is initial and terminal.

Prop Initial, terminal and zero object, if exist, must be unique up to isomorphisms.

Proof. Exercise.

Example. In Set, ϕ is initial and $\{*\}$ is terminal.

Example. In Grp, $\{1\}$ is zero object.

Example. In Mod_R, $\{0\}$ is zero object.

3. Product and coproduct as terminal and initial object.

- Let $\mathcal{C}$ be a category, $A_1, A_2 \in \mathrm{Ob}\,\mathcal{C}$. Define

$$\mathrm{Ob}(\mathcal{C}/\{A_1, A_2\}) = \{(X, f_1, f_2) \mid X \in \mathrm{ob}\,\mathcal{C},\ f_i \in \mathrm{Hom}(X, A_i), i = 1, 2\}.$$

$$\mathrm{Hom}_{\mathcal{C}/\{A_1, A_2\}}((X, f_1, f_2), (Y, g_1, g_2)) = \{h \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid g_i h = f_i,\ i = 1, 2\}.$$

$$\begin{tikzcd}
 & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "h"] & \\
A_1 & & A_2 \\
 & Y \arrow[ul, "g_1"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

$\mathcal{C}/\{A_1, A_2\}$ is a category.

Prop. Terminal object in $\mathcal{C}/\{A_1, A_2\}$ is product of A_1, A_2 in $\mathcal{C}$.

Proof. Terminal (P, π_1, π_2)

For any (B, f_1, f_2), there exists unique f such that

$$\begin{tikzcd}
 & B \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\exists! f"] & \\
A_1 & & A_2 \\
 & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

- Similarly, we can define $\mathcal{C}/\{A_1, A_2\}$ for coproduct

$$\mathrm{Ob}\,\mathcal{C}/\{A_1,A_2\} = \{ (X, f_1, f_2) \mid X \in \mathrm{Ob}\,\mathcal{C},\ f_i \in \mathrm{Hom}(A_i, X),\ i=1.2\}.$$

$$\mathrm{Hom}_{\mathcal{C}/\{A_1,A_2\}}((X,f_1,f_2),(Y,g_1,g_2)) = \{ h \in \mathrm{Hom}(X,Y) \mid hf_i = g_i,\ i=1,2\}.$$

$$\begin{tikzcd} & X \arrow[dd, "h"] & \\ A_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & A_2 \arrow[ul, "f_2"'] \arrow[dl, "g_2"] \\ & Y & \end{tikzcd}$$

Prop In above $\mathcal{C}/\{A_1, A_2\}$, the initial object is coproduct of A_1 and A_2 in the category $\mathcal{C}$.

Proof. Exercise.

4. Free object as initial object.

For concrete category $\mathcal{C}$ and a set X, we could define a category $\mathrm{Hom}(X, \mathcal{C})$.

- $\mathrm{Ob}\,\mathrm{Hom}(X,\mathcal{C}) = \bigcup_{A \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X, A)$
- Map between $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$ is

$$\begin{tikzcd} & A \arrow[dd, "h"] \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

Prop In $\mathrm{Hom}(X, \mathcal{C})$, an initial object is a free object over X in $\mathcal{C}$.

Proof. Exercise.

5. Pull-back

Consider a category $\mathcal{C}$ and two maps with the same codomain

$$\begin{tikzcd} A_1 \arrow[dr, "\varphi_1"] & \\ & C \\ A_2 \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

we define a category $\mathcal{D}$ as follows:

- $\mathrm{Ob}\,\mathcal{D} = \{ (X, f_1, f_2) \mid X \in \mathrm{Ob}\,\mathcal{C},\ f_i \in \mathrm{Hom}(X, A_i),\ i=1,2,\ \varphi_1 f_1 = \varphi_2 f_2 \}$

$$\begin{tikzcd}
 & A_1 \arrow[dr, "\varphi_1"] & \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & C \\
 & A_2 \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

- $\mathrm{Hom}_{\mathcal{D}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid g_i h = f_i,\ i=1,2 \}$.

$$\begin{tikzcd}
 & & A_1 \arrow[dr, "\varphi_1"] & \\
X \arrow[r, "h"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & Y \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & C \\
 & & A_2 \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

In this category, a terminal object (if exist) is called a pull-back of (φ_1, φ_2).

In other words, a pull-back is (Z, p_1, p_2) such that for any (X, f_1, f_2) satisfying $\varphi_1 f_1 = \varphi_2 f_2$, there exists unique $h: X \longrightarrow Z$ s.t. $p_i h = f_i$, $i=1,2$

$$\begin{tikzcd}
 & & A_1 \arrow[dr, "\varphi_1"] & \\
X \arrow[r, dashed, "\exists! h"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & Z \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & C \\
 & & A_2 \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

This is also called fiber product $A_1 \times_C A_2$.

6. Push-out

Consider a category $\mathcal{C}$ and two maps with the same domain

$$\begin{tikzcd}
 & A_1 \\
C \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\
 & A_2
\end{tikzcd}$$

We define a category $\mathcal{D}$ as follows

- $\mathrm{Ob}\,\mathcal{D} = \{ (X, f_1, f_2) \mid X \in \mathrm{Ob}\,\mathcal{C},\ f_i \in \mathrm{Hom}(A_i, X),\ i=1,2,\ f_1\psi_1 = f_2\psi_2 \}$

$$\begin{tikzcd}
& A_1 \arrow[dr, "f_1"] & \\
C \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & X \\
& A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

- $\mathrm{Hom}_{\mathcal{D}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid g_i = hf_i,\ i=1,2 \}$

$$\begin{tikzcd}
& A_1 \arrow[dr, "f_1"] \arrow[drr, bend left, "g_1"] & & \\
C \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & X \arrow[r, "h"] & Y \\
& A_2 \arrow[ur, "f_2"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

The initial object (W, q_1, q_2) in $\mathcal{D}$ is called push-out of ψ_1 and ψ_2.

In other words, a push-out (W, q_1, q_2) satisfies that: for any (X, f_1, f_2) with $f_1\psi_1 = f_2\psi_2$, there is a unique $h: W \to X$ such that $hq_i = f_i$.

$$\begin{tikzcd}
& A_1 \arrow[dr, "q_1"] \arrow[drr, bend left, "f_1"] & & \\
C \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & W \arrow[r, dashed, "\exists!\, h"] & X \\
& A_2 \arrow[ur, "q_2"'] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

7. Tensor product as initial objects

Consider module category Mod_R, fix $A, B \in \mathrm{Mod}_R$, we define a category $\mathcal{B}(A, B)$ as follows:

- $\mathrm{Ob}\,\mathcal{B}(A,B) = \{ \text{bilinear } f: A \times B \to C,\ C \in \mathrm{Mod}_R \}$
- $\mathrm{Hom}_{\mathcal{B}(A,B)}(f, g) = \{ h \in \mathrm{Hom}_{\mathrm{Mod}_R}(C, D) \mid g = hf \}$.

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[dr, "g"'] & C \arrow[d, "h"] \\ & D \end{tikzcd}$$

Then ~~tensor~~ product $A \otimes_R B$ is an initial object in $\mathcal{B}(A, B)$.

$$\begin{tikzcd} A \times B \arrow[r, "\otimes"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, dashed, "\exists! \tilde{f}"] \\ & X \end{tikzcd}$$