

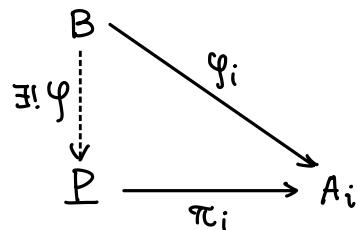
# Chapter 2 Categories

## §2.3 Products, Coproducts, and Universal Constructions

- Product
- Coproduct
- Free objects, initial and terminal objects, pull-back, push-out.

### (I) Product

Def 3.1 For  $\{A_i\}_{i \in I}$  in a given category  $\mathcal{C}$ , the product is defined as  $(P, \{\pi_i\})$  with  $P \in \text{Ob } \mathcal{C}$  and  $\pi_i: P \rightarrow A_i$  a family of maps such that for any  $(B, \{\varphi_i\})$  with  $\varphi_i: B \rightarrow A_i$ , there exists a unique  $\varphi: B \rightarrow P$  s.t.  $\pi_i \circ \varphi = \varphi_i$ ,  $\forall i \in I$



We denote the product as  $\prod_{i \in I} A_i$ .

Prop 3.1 If  $(P, \{\pi_i\})$  and  $(Q, \{\psi_i\})$  are both product of  $\{A_i\}_{i \in I}$ , then they are isomorphic.

*Proof.* For each product, we have

$$\begin{array}{ccc} P & \xrightarrow{\pi_i} & A_i \\ \exists! \psi \downarrow & \nearrow \psi_i & \\ Q & & \end{array} \quad \begin{array}{ccc} Q & \xrightarrow{\psi_i} & A_i \\ \exists! \pi \downarrow & \nearrow \pi_i & \\ P & & \end{array}$$

Then

$$\begin{array}{ccc} P & \xrightarrow{\pi_i} & A_i \\ \psi \circ \pi \downarrow & \nearrow \pi_i & \\ P & & \end{array} \quad \begin{array}{ccc} P & \xrightarrow{\pi_i} & A_i \\ \text{id}_P \downarrow & \nearrow \pi_i & \\ P & & \end{array}$$

Uniqueness implies  $\psi \circ \pi = \text{id}$ . Similarly  $\pi \circ \psi = \text{id}$ .

Remark. Generally, for  $\{A_i\}_{i \in I} \subseteq \text{Ob } \mathcal{C}$ , there may not exist a product.

But for Abelian category, the product and coproduct always exist.

## (II) Coproduct

Coproduct is dual concept to product.

Def 3.2 For  $\{A_i\}_{i \in I} \subseteq \text{Ob } \mathcal{C}$ , its coproduct is defined as  $(S, \{l_i : A_i \rightarrow S\})$

which satisfies: for any  $(B, \{\psi_i : A_i \rightarrow B\})$ , there exists a unique  $\psi$  s.t.

$$\psi l_i = \psi_i :$$

$$\begin{array}{ccc} A_i & \xrightarrow{l_i} & S \\ & \searrow \psi_i & \downarrow \exists! \psi \\ & & B \end{array}$$

We denote the coproduct as  $\coprod_{i \in I} A_i$ .

Prop 3.2. If  $(S, \{l_i\})$ ,  $(S', \{l'_i\})$  are both coproducts of  $\{A_i\}_{i \in I}$ , then they are isomorphic.

Proof. Similar to Prop 3.1

Example 3.1. In module category  $\text{Mod}_R$ , product and coproduct are direct product and direct sum.

Example 3.2. In  $\text{Set}$ : (1) product = Cartesian product

(2) coproduct = disjoint union

## (III) Some special objects.

1. Free object.

Def 3.3 A concrete category  $\mathcal{C}$  is a category that is equipped with a (faithful) functor to the  $\text{Set}$  category. More precisely:

(1) Every object  $A$  is assigned with a set  $\mathcal{G}(A)$ .

(2) Every map  $A \xrightarrow{f} B$  is assigned with a set map  $\mathcal{G}(A) \rightarrow \mathcal{G}(B)$ .

$$(3) \quad \text{id}_A = \text{id}_{\mathcal{C}(A)}$$

$$(4) \quad \mathcal{C}(f \circ g) = \mathcal{C}(f) \circ \mathcal{C}(g)$$

$$A \xrightarrow{g} B \xrightarrow{f} \quad \text{and} \quad \mathcal{C}(A) \xrightarrow{\mathcal{C}(f)} \mathcal{C}(B) \xrightarrow{\mathcal{C}(g)} \mathcal{C}(C)$$

Remark. In concrete category, we can regard object as a set equipped with some additional structure.

Example:  $\text{Grp}$ ,  $\text{Ring}$ ,  $\text{Vect}$ ,  $\text{Mod}_R$  are all concrete categories.

Recall that for free module  $M$  and its basis  $X$ , consider the inclusion  $l: X \hookrightarrow M$ , then for any module  $N$  and set map  $f: X \rightarrow N$ , there exists unique  $\tilde{f}$  s.t.  $f = \tilde{f} \circ l$ .

$$\begin{array}{ccc} X & \xleftarrow{l} & M \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & N \end{array}$$

This inspire the following definition of free objects.

Def 3.4 Let  $V$  be an object of some concrete category  $\mathcal{C}$ ,  $i: X \rightarrow V$  is a set map. If for any object  $A \in \text{Ob } \mathcal{C}$  and set map  $f: X \rightarrow A$ , there is a unique  $\tilde{f}: V \rightarrow A$  s.t.  $f = \tilde{f} \circ i$ , then  $V$  is called a free object over  $X$ .

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & A \end{array}$$

Example. free module is free object in  $\text{Mod}_R$ .

Prop 3.3 Let  $\mathcal{C}$  be a concrete category and  $V$  be a free object over  $X$ ,  $V'$  be a free object over  $X'$ , if  $|X| = |X'|$ , then  $V \cong V'$ .

Proof. Exercise.

## 2. Initial and terminal object.

Def 3.5 For a category  $\mathcal{C}$

means  $\uparrow$   $A \xrightarrow{f} B$  exists & unique

- (1)  $A$  is called an initial object if for any  $B$ ,  $\# \text{Hom}(A, B) = 1$ .
- (2)  $A$  is called a terminal object if for any  $B$ ,  $\# \text{Hom}(B, A) = 1$ .
- (3)  $A$  is called a zero or null object if it is initial and terminal.

Prop Initial, terminal and zero object, if exist, must be unique up to isomorphisms.

Proof. Exercise.

Example. In  $\text{Set}$ ,  $\emptyset$  is initial and  $\{\ast\}$  is terminal.

Example. In  $\text{Grp}$ ,  $\{1\}$  is zero object.

Example. In  $\text{Mod}_{\mathbb{R}}$ ,  $\{0\}$  is zero object.

## 3. Product and coproduct as terminal and initial object.

- Let  $\mathcal{C}$  be a category,  $A_1, A_2 \in \text{Ob } \mathcal{C}$ . Define

$$\text{Ob}(\mathcal{C}/\{A_1, A_2\}) = \{X, f_1, f_2 \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}(X, A_i), i=1,2\}.$$

$$\text{Hom}_{\mathcal{C}/\{A_1, A_2\}}((X, f_1, f_2), (Y, g_1, g_2)) = \{h \in \text{Hom}(X, Y) \mid g_i h = f_i, i=1,2\}.$$

$$\begin{array}{ccccc} & & X & & \\ & \swarrow f_1 & \downarrow h & \searrow f_2 & \\ A_1 & & Y & & A_2 \\ & \uparrow g_1 & & \downarrow g_2 & \end{array}$$

$\mathcal{C}/\{A_1, A_2\}$  is a category.

Prop. Terminal object in  $\mathcal{C}/\{A_1, A_2\}$  is product of  $A_1, A_2$  in  $\mathcal{C}$ .

Proof. Terminal  $(P, \pi_1, \pi_2)$

For any  $(B, f_1, f_2)$ , there exists unique  $f$  such that

$$\begin{array}{ccccc} & & B & & \\ & \swarrow f_1 & \downarrow \exists! f & \searrow f_2 & \\ A_1 & & P & & A_2 \\ & \uparrow \pi_1 & & \downarrow & \searrow \pi_2 \\ & & P & & \end{array}$$

- Similarly, we can define  $\mathcal{C}/\{A_1, A_2\}$  for coproduct

$$\begin{aligned} \text{ob } \mathcal{C}/\{A_1, A_2\} &= \{ (X, f_1, f_2) \mid X \in \text{ob } \mathcal{C}, f_i \in \text{Hom}(A_i, X), i=1, 2 \}. \\ \text{Hom}_{\mathcal{C}/\{A_1, A_2\}}((X, f_1, f_2), (Y, g_1, g_2)) &= \{ h \in \text{Hom}(X, Y) \mid h f_i = g_i, i=1, 2 \}. \end{aligned}$$

$$\begin{array}{ccccc} & & X & & \\ & f_1 \nearrow & \downarrow & \swarrow f_2 & \\ A_1 & & h & & A_2 \\ & g_1 \searrow & \downarrow & \swarrow g_2 & \\ & & Y & & \end{array}$$

Prop In above  $\mathcal{C}/\{A_1, A_2\}$ , the initial object is coproduct of  $A_1$  and  $A_2$  in the category  $\mathcal{C}$ .

Proof. Exercise.

#### 4. Free object as initial object.

For concrete category  $\mathcal{C}$  and a set  $X$ , we could define a category  $\text{Hom}(X, \mathcal{C})$ .

- $\text{ob } \text{Hom}(X, \mathcal{C}) = \bigcup_{A \in \text{ob } \mathcal{C}} \text{Hom}_{\mathcal{C}}(X, A)$
- Map between  $X \xrightarrow{f} A$  and  $X \xrightarrow{g} B$  is

$$\begin{array}{ccc} & f \nearrow & A \\ X & \downarrow h & \\ & g \searrow & B \end{array}$$

Prop In  $\text{Hom}(X, \mathcal{C})$ , an initial object is a free object over  $X$  in  $\mathcal{C}$ .

Proof. Exercise.

#### 5. Pull-back

Consider a category  $\mathcal{C}$  and two maps with the same codomain

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & C \\ A_2 & \xrightarrow{\varphi_2} & \end{array}$$

we define a category  $\mathcal{D}$  as follows:

- $\text{Ob } \mathcal{D} = \{ (X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}(X, A_i), i=1,2, g_1 f_1 = g_2 f_2 \}$

$$\begin{array}{ccc} & f_1 & \\ X & \nearrow & \downarrow \varphi_1 \\ & A_1 & \\ & f_2 & \searrow \\ & A_2 & \downarrow \varphi_2 \\ & & C \end{array}$$

- $\text{Hom}_{\mathcal{D}}(X, f_1, f_2), (Y, g_1, g_2) = \{ h \in \text{Hom}_{\mathcal{C}}(X, Y) \mid g_i h = f_i, i=1,2 \}$ .

$$\begin{array}{ccccc} & f_1 & & & \\ & \curvearrowright & & & \\ X & \xrightarrow{h} & Y & \xrightarrow{g_1} & A_1 \xrightarrow{\varphi_1} C \\ & f_2 & \curvearrowright & g_2 & \searrow \\ & & & & A_2 \xrightarrow{\varphi_2} C \end{array}$$

In this category, a terminal object (if exist) is called a pull-back of  $(\varphi_1, \varphi_2)$ .

In other words, a pull-back is  $(Z, p_1, p_2)$  such that for any  $(X, f_1, f_2)$  satisfying  $g_i f_1 = g_2 f_2$ , there exists unique  $h: X \rightarrow Z$  s.t.  $p_i h = f_i, i=1,2$

$$\begin{array}{ccccc} & f_1 & & & \\ & \curvearrowright & & & \\ X & \xrightarrow{\exists! h} & Z & \xrightarrow{p_1} & A_1 \xrightarrow{\varphi_1} C \\ & f_2 & \curvearrowright & p_2 & \searrow \\ & & & & A_2 \xrightarrow{\varphi_2} C \end{array}$$

This is also called fiber product  $A_1 \times_C A_2$ .

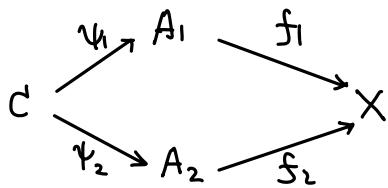
## 6. Push-out

Consider a category  $\mathcal{C}$  and two maps with the same domain

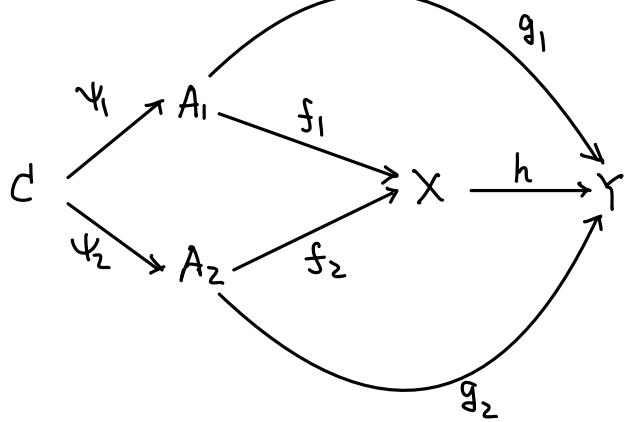
$$\begin{array}{ccc} & \varphi_1 & \\ C & \swarrow & \downarrow \\ & \varphi_2 & \searrow \\ & & A_2 \end{array}$$

We define a category  $\mathcal{D}$  as follows

- $\text{Ob } \mathcal{D} = \{ (X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}(A_i, X), i=1,2, f_1 \psi_i = f_2 \psi_i \}$

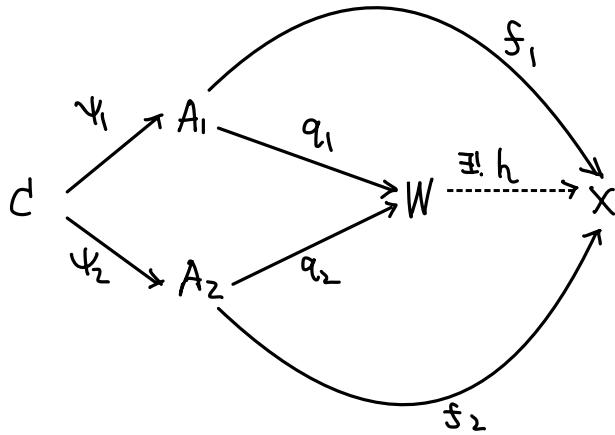


- $\text{Hom}_{\mathcal{D}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \text{Hom}_{\mathcal{C}}(X, Y) \mid g_i = h f_i, i=1,2 \}$



The initial object  $(W, q_1, q_2)$  in  $\mathcal{D}$  is called push-out of  $\psi_1$  and  $\psi_2$ .

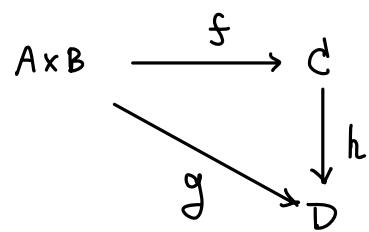
In other words, a push-out  $(W, q_1, q_2)$  satisfies that: for any  $(X, f_1, f_2)$  with  $f_1 \psi_1 = f_2 \psi_2$ , there is a unique  $h: W \rightarrow X$  such that  $h q_i = f_i$ .



## 7. Tensor product as initial objects

Consider module category  $\text{Mod}_R$ , fix  $A, B \in \text{Mod}_R$ , we define a category  $\mathcal{B}(A, B)$  as follows:

- $\text{Ob } \mathcal{B}(A, B) = \{ \text{bilinear } f: A \times B \rightarrow C, C \in \text{Mod}_R \}$
- $\text{Hom}_{\mathcal{B}(A, B)}(f, g) = \{ h \in \text{Hom}_{\text{Mod}_R}(C, D) \mid g = h f \}$ .



Then tensor product  $A \otimes_R B$  is an initial object in  $\mathcal{B}(A, B)$ .

